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Interim efficiency with MEU-preferences [☆]

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Abstract

Recently Kajii and Ui (2009) proposed to characterize interim efficient allocations in an exchange economy under asymmetric information when uncertainty is represented by multiple posteriors. When agents have Bewley's incomplete preferences, Kajii and Ui (2009) proposed a necessary and sufficient condition on the set of posteriors. However, when agents have Gilboa–Schmeidler's MaxMin expected utility preferences, Kajii and Ui (2009) only propose a sufficient condition.

The objective of this paper is to complete Kajii and Ui's work by proposing a necessary and sufficient condition for interim efficiency for various models of ambiguity aversion and in particular MaxMin expected utility. Our proof is based on a direct application of some results proposed by Rigotti, Shannon, and Strzalecki (2008).

JEL Classification: D81; D82; D84

Key words: Interim efficiency, Multiple priors and posteriors, MaxMin expected utility, No-trade

1. Introduction

We borrow from Kajii and Ui (2009) the following model of an exchange economy with a single good and a finite set of possible states of nature. Finitely many agents exchange contingent contracts. There are two stages: ex-ante each agent's perception of uncertainty is represented by a family of priors; at the interim stage each agent receives a private signal about which states will not occur and his interim perception of uncertainty is then represented by a family of posteriors. Each agent is endowed with a concave utility index function from which he derives either Bewley's incomplete preferences or Gilboa–Schmeidler's MaxMin expected utility preferences. The set of priors induces preferences at the ex-ante stage (before agents receive their private signal) and the set of posteriors induces preferences in the interim stage depending on the private signal agents receive.

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It is well-known that ex-ante efficiency can be characterized through a necessary and sufficient condition imposed on agents' priors. Bewley (2001) and Rigotti and Shannon (2005) characterized ex-ante efficiency for agents with Bewley's incomplete preferences.¹ Billot, Chateauneuf, Gilboa, and Tallon (2000) and Rigotti, Shannon, and Strzalecki (2008) characterized ex-ante efficiency for agents with Gilboa–Schmeidler's MaxMin expected utility preferences.²

In the standard Bayesian models, Morris (1994) and Feinberg (2000) provided a characterization of interim efficiency in terms of agents' posteriors. Kajii and Ui (2009) proposed to address the same question when agents have multiple posteriors. They identified a key concept, called the *compatible prior set*, that plays a crucial role in their analysis. The compatible prior set of an agent is the collection of all probability measures which, conditional to each private signal, coincides with a posterior.³ When agents have Bewley's incomplete preferences, Kajii and Ui (2009) succeeded to characterize interim efficiency by providing a necessary and sufficient condition in terms of compatible prior sets. For the particular case of linear utility index functions, they proved that an allocation is interim efficient if and only if the compatible prior sets of all agents have a non-empty intersection.

When agents have Gilboa–Schmeidler's MaxMin expected utility preferences, Kajii and Ui (2009) proposed a condition that is only sufficient. The objective of this paper is to show that it is possible to find a necessary and sufficient condition for interim efficiency when agents have Gilboa–Schmeidler's MaxMin expected utility preferences. The condition that we propose is closely related to the one introduced by Kajii and Ui (2009). Actually we show that the concept of compatible prior set is central not only for models where agents have Bewley's incomplete preferences or Gilboa–Schmeidler's MaxMin expected utility preferences, but also for any model with general convex preferences. More precisely, we provide a general necessary and sufficient condition for interim efficiency in terms of compatible priors associated to interim subjective beliefs as introduced by Rigotti, Shannon, and Strzalecki (2008). All the characterization results in Kajii and Ui (2009) follow as corollaries of our general characterization. In particular, having a complete characterization of ex-ante and interim efficiency, we can provide conditions under which there is no speculative trade as first studied by Milgrom and Stokey (1982) for standard Bayesian models.

The paper is organized as follows. Section 2 sets up the formal framework, notation and some preliminary definitions. The characterization results proposed by Kajii and Ui (2009) are presented in Section 3. Our necessary and sufficient condition for interim efficiency is stated and proved in Section 4. We illustrate in Section 5 how the results in Kajii and Ui (2009) can be deduced from our general characterization. Section 6 is devoted to no speculative trade and Section 7 shows how our results can be extended to encompass general convex preferences. We explore a slightly different concept of interim efficiency in Section 8 and Section 9 presents the corresponding no speculative trade results.

¹See also Dana (2000).

²See also Dana (1998), Samet (1998), Tallon (1998), Chateauneuf, Dana, and Tallon (2000), Dana (2002) and Dana (2004).

³Technically, the compatible prior set of an agent is the convex hull of all sets of the agent's posteriors.

2. Set up

We consider a model of an exchange economy \mathcal{E} under uncertainty with asymmetric information as presented in Kajii and Ui (2009). There is a finite set Ω of states. The set of all probability measures over Ω is denoted by $\text{Prob}(\Omega)$ and we let \mathcal{P} be the collection of all non-empty, convex and closed subsets of $\text{Prob}(\Omega)$. The expectation of a vector $x \in \mathbb{R}^\Omega$ under a probability measure $p \in \text{Prob}(\Omega)$ is denoted by $\mathbb{E}^p[x]$, i.e.,

$$\mathbb{E}^p[x] \equiv \sum_{\omega \in \Omega} p(\omega)x(\omega).$$

If P is a set in \mathcal{P} then we let

$$\mathbb{E}_P[x] \equiv \min_{p \in P} \mathbb{E}^p[x].$$

There is a finite set I of agents. Each agent i 's information is characterized by a partition Π^i of Ω . Any event $\pi \in \Pi^i$ can be interpreted as a signal received by agent i at the interim stage. The unique event $\pi \in \Pi^i$ containing ω is denoted by $\Pi^i(\omega)$. The information is assumed to be correct in the sense that if the state of nature is ω , agent i knows that the *true* state does not belong to $\Omega \setminus \Pi^i(\omega)$ but cannot discern among the states in $\Pi^i(\omega)$ which one is the true state. Each agent i has a set of priors $P^i \in \mathcal{P}$ which represents his prior beliefs, and a set of posteriors $\Phi^i(\pi) \in \mathcal{P}$ for each signal $\pi \in \Pi^i$, which represents his posterior beliefs after observing π . The collection of posteriors $\{\Phi^i(\pi)\}_{\pi \in \Pi^i}$ is denoted by Φ^i .

Assumption 1. For every agent i and every signal $\pi \in \Pi^i$,

- (a) there exists at least one prior $p \in P^i$ such that $p(\pi) > 0$;
- (b) every posterior $p \in \Phi^i(\pi)$ satisfies $p(\pi) = 1$.

For notational convenience, given a subset $\pi \subset \Omega$, we denote by $\mathcal{P}(\pi)$ the subset of \mathcal{P} defined as follows: a set $P \in \mathcal{P}$ belongs to $\mathcal{P}(\pi)$ if and only if the support of any probability in P is a subset of π , i.e.,

$$\forall p \in P, \quad p(\pi) = 1.$$

Observe that for every agent i and every interim signal $\pi \in \Pi^i$, the set of posteriors $\Phi^i(\pi)$ belongs to $\mathcal{P}(\pi)$.

There is a single good in the economy, and agent i has a concave, strictly increasing and continuous differentiable utility index function $u^i : [0, \infty) \rightarrow \mathbb{R}$ which induces MaxMin expected utility preferences as defined by Gilboa and Schmeidler (1989).

Definition 1. Agent i (strictly) prefers the contingent consumption bundle $y \in \mathbb{R}_+^\Omega$ to $x \in \mathbb{R}_+^\Omega$ at the *ex-ante* stage if

$$\mathbb{E}_{P^i}[u^i(y)] > \mathbb{E}_{P^i}[u^i(x)].$$

The set of all contingent consumption bundles that are (strictly) preferred to x at the *ex-ante* stage is denoted by $\text{Pref}_\Omega^i(x)$.

Similarly, we can define agent i 's preference relation at the interim stage.

Definition 2. Agent i (strictly) prefers the contingent consumption bundle $y \in \mathbb{R}_+^\Omega$ to $x \in \mathbb{R}_+^\Omega$ at the *interim* stage with private information $\pi \in \Pi^i$ if

$$\mathbb{E}_{\Phi^i(\pi)}[u^i(y)] > \mathbb{E}_{\Phi^i(\pi)}[u^i(x)].$$

The set of all contingent consumption bundles that are (strictly) preferred to x at the interim stage with private information $\pi \in \Pi^i$ is denoted by $\text{Pref}_\pi^i(x)$.

An allocation x is a family $x = (x^i)_{i \in I}$ where x^i is a vector in \mathbb{R}_+^Ω representing a contingent consumption bundle. We fix from now on an allocation $e = (e^i)_{i \in I}$ where e^i can be interpreted as the current endowment of agent i .

Assumption 2. For each agent i , the contingent consumption bundle e^i is interior in the sense that

$$\forall \omega \in \Omega, \quad e^i(\omega) > 0.$$

A family $t = (t^i)_{i \in I}$ where t^i is a vector in \mathbb{R}^Ω is called a feasible trade (from the allocation e) if

$$\forall i \in I, \quad e^i + t^i \in \mathbb{R}_+^\Omega \quad \text{and} \quad \sum_{i \in I} t^i = 0.$$

Each vector t^i correspond to the net trade of agent i .

We follow Kajii and Ui (2009) and introduce the concepts of *ex-ante* and *interim* efficiency.

Definition 3. The allocation e is

- *ex-ante efficient* if there does not exist a feasible trade t such that each agent i prefers at the ex-ante stage the contingent consumption $e^i + t^i$ to e^i ;
- *interim efficient* if there does not exist a feasible trade t such that each agent i prefers at every interim stage $\pi \in \Pi^i$ the contingent consumption $e^i + t^i$ to e^i .

In other words, the allocation e is not ex-ante efficient if and only if there exists a feasible trade t such that

$$\forall i \in I, \quad e^i + t^i \in \text{Pref}_\Omega^i(e^i).$$

The allocation e is not interim efficient if and only if there exists a feasible trade t such that

$$\forall i \in I, \quad \forall \pi \in \Pi^i, \quad e^i + t^i \in \text{Pref}_\pi^i(e^i).$$

In order to provide a characterization of efficiency in terms of primitives it is important to characterize the set of net trades t^i such that the associated contingent consumption $e^i + t^i$ is strictly preferred to the initial endowment e^i when agents have MEU-preferences. For that purpose, we need to introduce the concept of active belief.

Definition 4. Fix an agent i and a set $Q \in \mathcal{P}$ of beliefs. We denote by $\text{Act}^i(Q)$ the set of beliefs $p \in Q$ that minimize $\mathbb{E}^p[u^i(e^i)]$ over Q , i.e.,

$$\text{Act}^i(Q) \equiv \text{argmin}\{\mathbb{E}^p[u^i(e^i)] : p \in Q\}.$$

Any belief in $\text{Act}^i(Q)$ is called an *active belief* in Q at e^i .

Since we allow for risk-averse agents, we also need to introduce the concept of risk-adjusted belief.

Definition 5. Fix an agent i and a belief $p \in \text{Prob}(\Omega)$. The *risk-adjusted* belief $\text{RA}^i(p)$ is the probability measure in $\text{Prob}(\Omega)$ defined by⁴

$$\forall \omega \in \Omega, \quad \text{RA}^i(p)(\omega) \equiv \frac{p(\omega) \nabla u^i(e^i(\omega))}{\mathbb{E}^p[\nabla u^i(e^i)]}.$$

Given a set $Q \in \mathcal{P}$ of beliefs, we denote by $\text{RA}^i(Q)$ the set of risk-adjusted beliefs defined by

$$\text{RA}^i(Q) \equiv \bigcup_{q \in Q} \{\text{RA}^i(q)\}.$$

Adapting the arguments in Rigotti, Shannon, and Strzalecki (2008) we can prove the following lemma. This is the crucial technical result of this paper. We provide a detailed proof for more general preferences in Section 7.

Lemma 1. Fix an agent i , a set of beliefs $Q \in \mathcal{P}$ and a net trade $t^i \in \mathbb{R}^\Omega$ such that $e^i + t^i \geq 0$.

- If $e^i + t^i$ satisfies $\mathbb{E}_Q[u^i(e^i + t^i)] > \mathbb{E}_Q[u^i(e^i)]$ then

$$\forall p \in \text{RA}^i \circ \text{Act}^i(Q), \quad \mathbb{E}^p[t^i] > 0. \quad (1)$$

- Reciprocally, if t^i is such that (1) is satisfied then there exists $\varepsilon > 0$ small enough such that $\mathbb{E}_Q[u^i(e^i + \eta t^i)] > \mathbb{E}_Q[u^i(e^i)]$ for every $\eta \in (0, \varepsilon)$.

A direct consequence of this lemma and the fundamental theorem of welfare economics is the following characterization of ex-ante efficiency.⁵

Proposition 1. The allocation e is ex-ante efficient if and only if

$$\bigcap_{i \in I} \text{RA}^i \circ \text{Act}^i(P^i) \neq \emptyset.$$

It is natural to investigate whether a similar characterization is possible for interim efficiency.

3. The characterization proposed by Kajii and Ui

Kajii and Ui (2009) introduced the key concept of compatible priors which is a natural way to construct a family of priors when starting from a family of posterior beliefs.

⁴If $f : [0, \infty) \rightarrow \mathbb{R}$ is differentiable on $(0, \infty)$, we denote by $\nabla f(\alpha)$ the differential of f at $\alpha > 0$.

⁵See Proposition 2 and Proposition 7 in Rigotti, Shannon, and Strzalecki (2008) or Proposition 5 Kajii and Ui (2009). We also refer to Dana (1998), Samet (1998), Tallon (1998), Chateauneuf, Dana, and Tallon (2000), Billot, Chateauneuf, Gilboa, and Tallon (2000), Dana (2002) and Dana (2004).

Definition 6. Fix an agent i and a family $Q = (Q(\pi))_{\pi \in \Pi^i}$ of posterior beliefs where for each interim signal $\pi \in \Pi^i$, the set $Q(\pi)$ belongs to $\mathcal{P}(\Omega)$. A probability $p \in \text{Prob}(\Omega)$ is said to be a Q -compatible prior if for every interim signal $\pi \in \Pi^i$, the conditional probability $p(\cdot|\pi)$, when it exists, belongs to the set $Q(\pi)$. The set of all Q -compatible priors is denoted by $\text{CP}^i(Q)$.

Kajii and Ui (2009) showed that the set of Q -compatible priors is actually the convex hull of the union of all sets $Q(\pi)$, i.e.,

$$\text{CP}^i(Q) = \text{co} \bigcup_{\pi \in \Pi^i} Q(\pi).$$

In particular the set $\text{CP}^i(Q)$ is non-empty, convex and closed, i.e., it belongs to \mathcal{P} .

Unfortunately, Kajii and Ui (2009) did not propose a “general” characterization of interim efficiency similar to the characterization given in Proposition 1 for ex-ante efficiency. They propose a sufficient condition and prove that this condition is necessary provided that the interim utility of the initial endowment is independent of the signal received, i.e., the following mapping

$$\pi \mapsto \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i)]$$

is constant over Π^i for every agent i .

Definition 7. The allocation e is said to have *constant interim utility* if for every agent i , the utility at an interim stage of the contingent consumption e^i is independent of the signal received, i.e.,

$$\forall i \in I, \quad \forall \pi, \sigma \in \Pi^i, \quad \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i)] = \mathbb{E}_{\Phi^i(\sigma)}[u^i(e^i)].$$

Kajii and Ui (2009) proved the following (partial) characterization.

Proposition 2.

(a) The allocation e is interim efficient if⁶

$$\bigcap_{i \in I} \text{RA}^i \circ \text{Act}^i \circ \text{CP}^i(\Phi^i) \neq \emptyset. \quad (2)$$

(b) If the allocation e has constant interim utility then condition (2) is also necessary.

In order to provide a necessary and sufficient condition for interim efficiency, Kajii and Ui (2009) introduced the concept of full insurance in the interim stage.

Definition 8. A contingent consumption bundle x has the *full-insurance property at the interim stage* for agent i if it is *privately measurable* in the sense that the restriction of x to each signal $\pi \in \Pi^i$ is constant.

Kajii and Ui (2009) proposed the following necessary and sufficient condition when the allocation e is privately measurable.

Proposition 3. Assume that the allocation e is privately measurable in the sense that for each agent i , the contingent consumption bundle e^i has the full-insurance property at the interim stage. Then the allocation e is interim efficient if and only if

$$\bigcap_{i \in I} \text{RA}^i \circ \text{CP}^i(\Phi^i) \neq \emptyset. \quad (3)$$

⁶We abuse notations by writing $\text{RA}^i \circ \text{Act}^i \circ \text{CP}^i(\Phi^i)$ instead of $\text{RA}^i \left[\text{Act}^i \left[\text{CP}^i(\Phi^i) \right] \right]$.

4. A necessary and sufficient condition for interim efficiency

We propose to improve the latter results by exhibiting a necessary and sufficient condition. For expositional reasons we abuse notations by writing $CP^i \circ RA^i \circ Act^i(\Phi^i)$ instead of

$$CP^i \left[\left(RA^i \left[Act^i(\Phi^i(\pi)) \right] \right)_{\pi \in \Pi^i} \right].$$

Theorem 1. *The allocation e is interim efficient if and only if*

$$\bigcap_{i \in I} CP^i \circ RA^i \circ Act^i(\Phi^i) \neq \emptyset. \quad (4)$$

In other words, the allocation e is interim efficient if and only if there exists a probability measure $q \in \text{Prob}(\Omega)$ and for each i , a family $r^i = (r_\pi^i)_{\pi \in \Pi^i}$ with r_π^i an active belief of $\Phi^i(\pi)$ at e^i such that

$$\forall i \in I, \quad q = \sum_{\pi \in \Pi^i} q(\pi) RA^i(r_\pi^i).$$

The proof will be a very simple consequence of Proposition 1. To see this, we need the following intermediate result.

Lemma 2. *The allocation e is interim efficient if and only if there does not exist a feasible trade t such that*

$$\forall i \in I, \quad \forall p \in \bigcup_{\pi \in \Pi^i} RA^i \circ Act^i(\Phi^i(\pi)), \quad \mathbb{E}^p[t^i] > 0. \quad (5)$$

Proof of Lemma 2. We first prove the “if” part. Assume that there does not exist a feasible trade satisfying (5) but the allocation e is not interim efficient. Then there exists a feasible trade t such that

$$\forall i \in I, \quad \forall \pi \in \Pi^i, \quad \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i + t^i)] > \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i)].$$

Following Lemma 1 we must have

$$\forall i \in I, \quad \forall \pi \in \Pi^i, \quad \forall p \in RA^i \circ Act^i(\Phi^i(\pi)), \quad \mathbb{E}^p[t^i] > 0.$$

This contradicts the assumption that there does not exist a feasible trade satisfying (5).

We now prove the “only if” part. Assume that the allocation e is interim efficient but there exists a feasible trade τ such that

$$\forall i \in I, \quad \forall p \in \bigcup_{\pi \in \Pi^i} RA^i \circ Act^i(\Phi^i(\pi)), \quad \mathbb{E}^p[\tau^i] > 0.$$

Fix an agent i and a signal $\pi \in \Pi^i$. We have

$$\forall p \in RA^i \circ Act^i(\Phi^i(\pi)), \quad \mathbb{E}^p[\tau^i] > 0.$$

It follows from Lemma 1 that there exists $\varepsilon_\pi^i > 0$ such that

$$\forall \eta \in (0, \varepsilon_\pi^i), \quad \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i + \eta \tau^i)] > \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i)].$$

We let $\varepsilon > 0$ be defined by

$$\varepsilon \equiv \min\{\varepsilon_\pi^i : i \in I \text{ and } \pi \in \Pi^i\}$$

and for each i , we pose $t^i = \varepsilon \tau^i$. The allocation $t = (t^i)_{i \in I}$ is a feasible trade such that

$$\forall i \in I, \quad \forall \pi \in \Pi^i, \quad e^i + t^i \in \text{Pref}_\pi^i(e^i)$$

which leads to a contradiction. \square

The proof of Theorem 1 follows from Lemma 2 and Proposition 1. We provide the straightforward details hereafter.

Proof of Theorem 1. We consider the *modified* economy $\widehat{\mathcal{E}}$ where

- each agent i is risk-neutral in the sense that his utility index \widehat{u}^i is linear, i.e., $\widehat{u}^i(c) = c$;
- each agent i 's set of priors \widehat{P}^i is defined by

$$\widehat{P}^i \equiv \text{co} \bigcup_{\pi \in \Pi^i} \text{RA}^i \circ \text{Act}^i(\Phi^i(\pi)).$$

According to Lemma 2, the allocation e is interim efficient for the economy \mathcal{E} if and only if it is ex-ante efficient for the economy $\widehat{\mathcal{E}}$. Observe that \widehat{P}^i coincides with $\text{CP}^i \circ \text{RA}^i \circ \text{Act}^i(\Phi^i)$. Applying Proposition 1, we obtain the desired result. \square

We provide hereafter an example where Theorem 1 can be applied but not the results in Kajii and Ui (2009).

Example 1. Consider an exchange economy with two risk-neutral agents $I = \{i_1, i_2\}$ and four states of nature $\Omega = \{\omega_1, \dots, \omega_4\}$.⁷ Agent i_1 may receive two signals $\Pi^{i_1} = \{a, b\}$ with $a = \{\omega_1, \omega_2\}$ and $b = \{\omega_3, \omega_4\}$. His posterior beliefs are given by

$$\Phi^{i_1}(a) = \{p \in \text{Prob}(\Omega) : p(\omega_1) + p(\omega_2) = 1 \text{ and } 1/4 \leq p(\omega_1) \leq 1/2\}$$

and

$$\Phi^{i_1}(b) = \{p \in \text{Prob}(\Omega) : p(\omega_3) + p(\omega_4) = 1 \text{ and } 1/4 \leq p(\omega_3) \leq 1/2\}.$$

Agent i_1 's contingent plan is $e^{i_1} = (1, 3, 3, 1)$. Agent i_2 has no private information, i.e., $\Pi^{i_2} = \{\Omega\}$ and his posterior beliefs are represented by a single probability

$$\Phi^{i_2}(\Omega) = \{(1/4, 1/4, 1/8, 3/8)\}.$$

Agent i_2 's contingent plan is any interior vector $e^{i_2} \in \mathbb{R}_{++}^\Omega$. We can compute the set of active priors for agent i_1 :

$$\text{Act}^{i_1}(\Phi^{i_1}(a)) = \{(1/2, 1/2, 0, 0)\} \text{ and } \text{Act}^{i_1}(\Phi^{i_1}(b)) = \{(0, 0, 1/4, 3/4)\}.$$

It follows that

$$\text{CP}^{i_1} \circ \text{Act}^{i_1}(\Phi^{i_1}) = \{p \in \text{Prob}(\Omega) : p(\omega_1) = p(\omega_2) \text{ and } p(\omega_4) = 3p(\omega_3)\}.$$

⁷The utility index function u^i of each agent i is defined by $u^i(c) = c$ for each $c \geq 0$.

Since the unique posterior belief of agent i_2 belongs to $CP^{i_1} \circ Act^{i_1}(\Phi^{i_1})$, we can apply Theorem 1 to conclude that the allocation e is interim efficient. Since the interim expected utility of agent i_1 is not constant, we cannot apply Proposition 6 in Kajii and Ui (2009).⁸ Neither can we apply Proposition 7 in Kajii and Ui (2009) since e^{i_1} does not have the full-insurance property at the interim stage.

5. Relation with the literature

In order to simplify the comparison between our characterization result and those presented in Kajii and Ui (2009), we propose to state some properties satisfied by the operators Act^i , CP^i and RA^i . The details of the proofs are postponed to Appendices.

Lemma 3. *For every agent i and every family $Q = (Q(\pi))_{\pi \in \Pi^i}$ of posterior beliefs $Q(\pi) \in \mathcal{P}(\pi)$, we have*

$$CP^i \circ RA^i(Q) = RA^i \circ CP^i(Q).$$

Remark 1. As a consequence of Theorem 1 and Lemma 4 we obtain the following equivalent characterization: The allocation e is interim efficient if and only if

$$\bigcap_{i \in I} RA^i \circ CP^i \circ Act^i(\Phi^i) \neq \emptyset. \quad (6)$$

Lemma 4. *For every agent i and every family $Q = (Q(\pi))_{\pi \in \Pi^i}$ of posterior beliefs $Q(\pi) \in \mathcal{P}(\pi)$, we always have*

$$Act^i \circ CP^i(Q) \subset CP^i \circ Act^i(Q).$$

The converse inclusion is true if e has constant interim utility.

The partial characterization proved by Kajii and Ui (2009) (and presented in Proposition 2) follows from our main characterization result (Theorem 1) and the two preceding lemmas. One may want to compare Proposition 3 with our general necessary and sufficient condition. The following characterization is a straightforward consequence of Theorem 1.

Proposition 4. Assume that the allocation e is privately measurable. Then the allocation e is interim efficient if and only if

$$\bigcap_{i \in I} CP^i(\Phi^i) \neq \emptyset. \quad (7)$$

Proof of Proposition 4. Assume that the allocation e is privately measurable. It is sufficient to prove that for each agent i and each private signal $\pi \in \Pi^i$, we have $RA^i \circ Act^i(\Phi^i(\pi)) = \Phi^i(\pi)$. Since e^i is constant on π , we denote by $e^i(\pi)$ its value. Observe that for each $r_\pi \in \Phi^i(\pi)$ we have $\mathbb{E}^{r_\pi}[u^i(e^i)] = u^i(e^i(\pi))$. In particular, any posterior belief is active, i.e., $Act^i[\Phi^i(\pi)] = \Phi^i(\pi)$.

⁸We can check that

$$\mathbb{E}_{\Phi^{i_1}(a)}[u^{i_1}(e^{i_1})] = 2 \quad \text{and} \quad \mathbb{E}_{\Phi^{i_1}(b)}[u^{i_1}(e^{i_1})] = 3/2.$$

Observe moreover that

$$Act^{i_1} \circ CP^{i_1}(\Phi^{i_1}) = \{(0, 0, 1/4, 3/4)\}.$$

Since the unique posterior belief of agent i_2 does not belong to $Act^{i_1} \circ CP^{i_1}(\Phi^{i_1})$, condition (7) in Kajii and Ui (2009) cannot be necessary.

We propose now to prove that there is no need to adjust for risk. This is very intuitive since there is no risk. More precisely, let κ_π be a probability measure in $\text{RA}^i(\Phi^i(\pi))$. There exists a posterior belief $r_\pi \in \Phi^i(\pi)$ such that

$$\forall \omega \in \Omega, \quad \kappa_\pi(\omega) = \frac{1}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]} \nabla u^i(e^i(\omega)) r_\pi(\omega).$$

Since e^i is constant on π , the function $\omega \mapsto \nabla u^i(e^i(\omega))$ is also constant on π . We denote by $\nabla u^i(e^i(\pi))$ its value. It follows that $\mathbb{E}^{r_\pi}[\nabla u^i(e^i)] = \nabla u^i(e^i(\pi))$ implying that $\kappa_\pi = r_\pi$. \square

Our necessary and sufficient condition (7) seems to be different from (3) the condition proposed by Kajii and Ui (2009). Actually, when the allocation e has the full-insurance property at the interim stage, there is no need to adjust Φ -compatible priors to risk.

Lemma 5. *If the allocation e is privately measurable then*

$$\forall i \in I, \quad \text{RA}^i \circ \text{CP}^i(\Phi^i) = \text{CP}^i(\Phi^i).$$

The proofs of Lemma 3, Lemma 4 and Lemma 5 can be found in Appendix A, Appendix B and Appendix C respectively.

6. No speculative trade

Following Kajii and Ui (2009) we propose to investigate under which conditions speculative trade is impossible.

Definition 9. We say that there is no *speculative trade* if ex-ante efficiency of the allocation e implies that it is also interim efficient.

As a direct consequence of Proposition 1 and Theorem 1 we get the following general characterization.

Proposition 5. There is no speculative trade if and only if

$$\bigcap_{i \in I} \text{RA}^i \circ \text{Act}^i(P^i) \neq \emptyset \implies \bigcap_{i \in I} \text{CP}^i \circ \text{RA}^i \circ \text{Act}^i(\Phi^i) \neq \emptyset. \quad (8)$$

The two general characterization results (Corollary 8 and Corollary 9) proposed by Kajii and Ui (2009) follow from the previous result together with Lemma 3, Lemma 4 and Lemma 5.

Proposition 6. Assume that the allocation e has constant interim utility. Then there is no speculative trade if and only if

$$\bigcap_{i \in I} \text{RA}^i \circ \text{Act}^i(P^i) \neq \emptyset \implies \bigcap_{i \in I} \text{RA}^i \circ \text{Act}^i \circ \text{CP}^i(\Phi^i) \neq \emptyset. \quad (9)$$

Assume that the allocation e is privately measurable. Then there is no speculative trade if and only if

$$\bigcap_{i \in I} \text{RA}^i \circ \text{Act}^i(P^i) \neq \emptyset \implies \bigcap_{i \in I} \text{CP}^i(\Phi^i) \neq \emptyset. \quad (10)$$

Remark 2. In (Kajii and Ui, 2009, Corollary 9), the condition (10) is replaced by

$$\bigcap_{i \in I} \text{RA}^i \circ \text{Act}^i(P^i) \neq \emptyset \implies \bigcap_{i \in I} \text{RA}^i \circ \text{CP}^i(\Phi^i) \neq \emptyset. \quad (11)$$

We proved (see Lemma 5) that when the allocation e is privately measurable then for every i , we have $\text{RA}^i \circ \text{CP}^i(\Phi^i) = \text{CP}^i(\Phi^i)$. This implies that both conditions (11) and (10) are equivalent.

The necessary and sufficient condition (8) imposes an abstract relation between the set of priors P^i and the family $\Phi^i = (\Phi^i(\pi))_{\pi \in \Pi^i}$ of posteriors. We propose to investigate sufficient conditions relating priors and posteriors to preclude speculative trade. A straightforward sufficient condition is proposed hereafter.

Proposition 7. Assume that for each agent i , for every active prior $p^i \in \text{Act}^i(P^i)$ and for every signal $\pi \in \Pi^i$ the condition probability $p^i(\cdot|\pi)$, when it exists, is an active posterior, i.e.,

$$\forall p^i \in \text{Act}^i(P^i), \quad \forall \pi \in \Pi^i, \quad p^i(\pi) > 0 \implies p^i(\cdot|\pi) \in \text{Act}^i(\Phi^i(\pi)). \quad (12)$$

Then there is no speculative trade.

Proof of Proposition 7. We only have to check that (8) is satisfied. Assume that there exists a probability $q \in \text{Prob}(\Omega)$ and for each agent i an active prior $p^i \in \text{Act}^i(P^i)$ such that

$$\forall i \in I, \quad q = \text{RA}^i(p^i).$$

Fix a state $\omega \in \Omega$ and an agent i . We denote by π the associated signal $\Pi^i(\omega)$. If $p^i(\pi) > 0$ then

$$\begin{aligned} q(\omega) &= \frac{1}{\mathbb{E}^{p^i}[\nabla u^i(e^i)]} \nabla u^i(e^i(\omega)) p^i(\omega) \\ &= \frac{p^i(\pi)}{\mathbb{E}^{p^i}[\nabla u^i(e^i)]} \nabla u^i(e^i(\omega)) p^i(\omega|\pi) \\ &= \lambda_\pi^i \text{RA}^i(p^i(\cdot|\pi))(\omega) \end{aligned}$$

where

$$\lambda_\pi^i \equiv \frac{p^i(\pi) \mathbb{E}^{p^i(\cdot|\pi)}[\nabla u^i(e^i)]}{\mathbb{E}^{p^i}[\nabla u^i(e^i)]}.$$

Since the family $(\lambda_\pi^i)_{\pi \in \Pi^i}$ belongs to $\text{Prob}(\Pi^i)$, it follows that for each agent i , the probability q belongs to the set $\text{CP}^i \circ \text{RA}^i \circ \text{Act}^i(\Phi^i)$. \square

Condition (12) is still an abstract relation between the set of priors and the family of posteriors. However, it is now simple to provide explicit conditions on the way agents “update” their beliefs to guarantee that no speculative trade is possible. Kajii and Ui (2009) used the concepts of *full Bayesian updating*. We consider a weaker concept.

Definition 10. We say that the set of posteriors Φ^i is *Bayesian consistent* with P^i if for every posterior belief $p \in P^i$ and for every private signal $\pi \in \Pi^i$ plausible according to p , i.e., $p(\pi) > 0$, the conditional probability $p(\cdot|\pi)$ is a possible posterior, i.e.,

$$\forall p \in P^i, \quad \forall \pi \in \Pi^i, \quad p(\pi) > 0 \implies p(\cdot|\pi) \in \Phi^i(\pi). \quad (13)$$

As a simple corollary of Proposition 7 we obtain the following no-trade result.

Corollary 1. *Assume that for each agent i the set of posteriors is Bayesian consistent with the set of priors. If the allocation e is privately measurable (or equivalently satisfies the full insurance property at the interim stage) then there is no speculative trade.*

This is a slight generalization of Proposition 10 in Kajii and Ui (2009) since we only assume that the set of posteriors is Bayesian consistent with priors, while Kajii and Ui (2009) assumed that posteriors are the full Bayesian updating of priors in the sense that for every agent i ,

$$\forall \pi \in \Pi^i, \quad \Phi^i(\pi) = \text{cl}\{p(\cdot|\pi) : p \in P^i \text{ and } p(\pi) > 0\}.$$

Proof of Corollary 1. We only have to check that (12) is satisfied. Fix an agent i , an active prior $p^i \in \text{Act}^i(P^i)$ and a private signal $\pi \in \Pi^i$ with $p^i(\pi) > 0$. Since P^i is Bayesian consistent with Φ^i , the conditional belief $p^i(\cdot|\pi)$ belongs to $\Phi^i(\pi)$. We should now prove that $p^i(\cdot|\pi)$ is active. Actually, a direct consequence of the measurability of e^i is that any prior is active, i.e., $\text{Act}^i(\Phi^i(\pi)) = \Phi^i(\pi)$. Indeed, for every posterior $r_\pi \in \Phi^i(\pi)$, we have

$$\mathbb{E}^{r_\pi}[u^i(e^i)] = u^i(e^i(\pi)).$$

□

We can also obtain the no-trade result of Epstein and Schneider (2003) and Wakai (2008) as a simple corollary of Proposition 7. These authors proved that if priors are rectangular sets with respect to posteriors, then MEU-preferences are dynamically consistent which in turn implies that there is no speculative trade.⁹ In order to recall the concept of a rectangular prior set, we introduce some notations. Fix a probability $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ in $\text{Prob}(\Pi^i)$ representing beliefs about the private signals of agent i . Let $r = (r_\pi)_{\pi \in \Pi^i}$ be a family of posteriors. The probability in $\text{Prob}(\Omega)$ defined by

$$\sum_{\pi \in \Pi^i} \lambda_\pi r_\pi$$

is denoted by $\lambda \otimes r$. If Λ^i is a set of probabilities in $\text{Prob}(\Pi^i)$, we denote by $\Lambda^i \otimes \Phi^i$ the set of all probabilities $\lambda \otimes r$ where $\lambda \in \Lambda^i$ and $r_\pi \in \Phi^i(\pi)$ for each $\pi \in \Pi^i$. Observe that the set $\text{CP}^i(\Phi^i)$ of Φ^i -compatible priors coincides with the set $\text{Prob}(\Pi^i) \otimes \Phi^i$.

Definition 11. The set of priors is *rectangular with respect to posteriors* (or equivalently P^i is Φ^i -rectangular) if there exists a set Λ^i of beliefs in $\text{Prob}(\Pi^i)$ about the realization of private signals such that

$$P^i = \Lambda^i \otimes \Phi^i$$

and such that for every private signal $\pi \in \Pi^i$, there exists $\lambda \in \Lambda^i$ such that $\lambda_\pi > 0$.

If $P^i = \Lambda^i \otimes \Phi^i$ then Λ^i must coincide with $P^i(\Pi^i)$ the set of all probabilities $(p(\pi))_{\pi \in \Pi^i}$ defined by all priors $p \in P^i$. Observe that if Φ^i is Bayesian consistent with P^i then we have

$$P^i \subset P^i(\Pi^i) \otimes \Phi^i.$$

⁹For the readers interested in dynamically consistent update rules for decision making under ambiguity, we refer to Hanany and Klibanoff (2007) and the literature therein.

When the inclusion is replaced by an equality, we obtain that P^i is Φ^i -rectangular. It is straightforward that check if the set of priors is rectangular with respect to posteriors then posteriors are the full Bayesian updating of priors.¹⁰

As a simple corollary of Proposition 7 we obtain the following no-trade result which corresponds to Proposition 11 in Kajii and Ui (2009).

Corollary 2. *If for each agent the set of priors is rectangular with respect to posteriors, then there is no speculative trade.*

Proof of Corollary 2. We only have to check that (12) is satisfied. Fix an agent i , an active prior $p^i \in \text{Act}^i(P^i)$ and a private signal $\pi \in \Pi^i$ with $p^i(\pi) > 0$. Since the set of priors P^i is Φ^i -rectangular, it is Bayesian consistent with Φ^i . Therefore, the conditional belief $p^i(\cdot|\pi)$ belongs to $\Phi^i(\pi)$. We should now prove that the posterior $p^i(\cdot|\pi)$ is active, i.e., belongs to $\text{Act}^i(\Phi^i(\pi))$. We know that p^i is an active prior, i.e.,

$$\forall q \in P^i, \quad \mathbb{E}^{p^i}[u^i(e^i)] \leq \mathbb{E}^q[u^i(e^i)]. \quad (14)$$

Fix an arbitrary posterior $r_\pi \in \Phi^i(\pi)$ and let q be the probability in $\text{Prob}(\Omega)$ defined by

$$q = p^i(\pi)r_\pi + \sum_{\sigma \neq \pi} p^i(\sigma)p(\cdot|\sigma).$$

Since the set of priors is Φ^i -rectangular, the probability q also belongs to P^i . Applying (14) we get

$$\mathbb{E}^{p^i(\cdot|\pi)}[u^i(e^i)] \leq \mathbb{E}^{r_\pi}[u^i(e^i)].$$

We have proved that $p^i(\cdot|\pi)$ is an active posterior. □

We propose hereafter an example where Proposition 7 can be applied but the results in Kajii and Ui (2009) cannot.

Example 2. Consider the economy described in Example 1. We now fix prior beliefs for both agents. Since agent i_2 has no private information, we assume that he has a unique prior belief which coincides with his posterior belief, i.e., $P^{i_2} = \{(1/4, 1/4, 1/8, 3/8)\}$. To describe the priors of agent i_1 , we propose the following parametrization of his posterior beliefs:

$$\forall \pi \in \Pi^{i_1} = \{a, b\}, \quad \Phi^{i_1}(\pi) = \{r_\pi^\theta : \theta \in \Theta\}$$

where $\Theta = [0, 1/4]$ and

$$r_a^\theta = (1/2 - \theta, 1/2 + \theta, 0, 0) \quad \text{and} \quad r_b^\theta = (0, 0, 1/4 + \theta, 3/4 - \theta).$$

The parameter θ can be interpreted as a scenario that agent i_1 considers plausible. This parameter describes agent i_1 's ambiguity in the sense that θ cannot be observed and agent i_1 has no beliefs about its realization. We assume that contingent to a scenario θ , agent i_1

¹⁰Actually we also have that Φ^i is the maximum likelihood updating of P^i in the sense that

$$\forall \pi \in \Pi^i, \quad \Phi^i(\pi) = \left\{ p(\cdot|\pi) : p \in \arg\max_{q \in P^i} q(\pi) \right\}.$$

believes he will receive the signal a with probability $\lambda_a^\theta \in (0, 1)$. We pose $\lambda_b^\theta = 1 - \lambda_a^\theta$ such that λ^θ belongs to $\text{Prob}(\Pi^{i_1})$. The prior beliefs of agent i_1 are constructed in the following way:

$$P^{i_1} = \{\lambda^\theta \otimes r^\theta : \theta \in \Theta\}.$$

For each plausible scenario θ , agent i_1 's prior belief is represented by the probability measure

$$p^\theta = \lambda^\theta \otimes r^\theta = (\lambda_a^\theta((1/2) - \theta), \lambda_a^\theta(1/2 + \theta), \lambda_b^\theta(1/4 + \theta), \lambda_b^\theta(3/4 - \theta)).$$

Observe that Φ^{i_1} is the full Bayesian updating of P^{i_1} in the sense that $\Phi^{i_1}(\pi) = \{p(\cdot|\pi) : p \in P^{i_1}\}$. However, P^{i_1} is not Φ^{i_1} -rectangular.

We assume that $\theta \mapsto \lambda_a^\theta$ is strictly increasing with $\lambda_a^0 = 1/2$.¹¹ Since

$$\mathbb{E}^{p^\theta} [u^{i_1}(e^{i_1})] = 2\theta + \frac{3}{2} + \frac{\lambda_a^\theta}{2}$$

it follows that the only active prior belief is p^0 , i.e., $\text{Act}^{i_1}(P^{i_1}) = \{\lambda^0 \otimes r^0\}$. Since p^0 coincides with the unique prior of agent i_2 , the allocation e is ex-ante efficient. Observe that

$$p^0(\cdot|a) = r_a^0 \in \text{Act}^{i_1}(\Phi^{i_1}(a)) \quad \text{and} \quad p^0(\cdot|b) = r_b^0 \in \text{Act}^{i_1}(\Phi^{i_1}(b))$$

implying that we can apply Proposition 7 to conclude that there is no speculative trade. Since the interim expected utility of agent i_1 is not constant, we cannot apply Corollary 8 in Kajii and Ui (2009). Neither can we apply Corollary 9 or Corollary 10 since e^{i_1} does not have the full-insurance property at the interim stage.

7. General convex preference and subjective beliefs

Until now we assumed that agents have MaxMin expected utility preferences. When uncertainty is represented by multiple priors and posteriors, there are other modelings of preferences: the incomplete preferences model of Bewley (2002), the convex Choquet model of Schmeidler (1989), the smooth second-order prior models of Klibanoff, Marinacci, and Mukerji (2005) and Nau (2006), the second-order expected utility model of Ergin and Gul (2009), the confidence preferences model of Chateauneuf and Faro (2006), the multiplier model of Hansen and Sargent (2001), and the variational preferences model of Maccheroni, Marinacci, and Rustichini (2006). We propose to follow the approach initiated by Rigotti, Shannon, and Strzalecki (2008) by considering a broad class of convex preferences which encompasses as special cases all the aforementioned models.

Each agent i has an ex-ante preference relation \succ_Ω^i on contingent consumption bundles in \mathbb{R}_+^Ω defining the correspondence $\text{Pref}_\Omega^i : \mathbb{R}_+^\Omega \rightarrow \mathbb{R}_+^\Omega$ of strictly preferred contingent consumption bundles:

$$\forall x \in \mathbb{R}_+^\Omega, \quad \text{Pref}_\Omega^i(x) \equiv \{y \in \mathbb{R}_+^\Omega : y \succ_\Omega^i x\}.$$

Similarly, for each possible interim signal $\pi \in \Pi^i$, agent i is endowed with an interim preference relation \succ_π^i on consumption bundles contingent to the information π , defining the correspondence $\text{Pref}_\pi^i : \mathbb{R}_+^\pi \rightarrow \mathbb{R}_+^\pi$ of strictly preferred contingent consumption bundles at the

¹¹ Take for example $\lambda^\theta = (1/2 + \theta, 1/2 - \theta)$ for every $\theta \in [0, 1/4]$.

interim stage. If x is a vector in \mathbb{R}^Ω and σ is a subset of Ω , we denote by $x|_\sigma$ the restriction of x to σ , i.e., $x|_\sigma$ is the vector in \mathbb{R}^σ defined by $(x|_\sigma)(\omega) = x(\omega)$ for each $\omega \in \sigma$.

Preference relations are assumed to satisfy the following properties

Assumption 3. For each agent i , for each $\sigma \in \{\Omega\} \cup \Pi^i$, the binary relation \succ_σ^i is

- (a) irreflexive, i.e., $x \notin \text{Pref}_\sigma^i(x)$ for all $x \in \mathbb{R}_+^\sigma$;
- (b) convex, i.e., the set $\text{Pref}_\sigma^i(x)$ is convex for all $x \in \mathbb{R}_+^\sigma$;
- (c) monotone, i.e., $x + h \in \text{Pref}_\sigma^i(x)$ for all $x, h \in \mathbb{R}_+^\sigma$ and h interior;¹²
- (d) continuous, i.e., the set $\text{Pref}_\sigma^i(x)$ is open in \mathbb{R}_+^σ .

Following Rigotti, Shannon, and Strzalecki (2008) we introduce the concepts of ex-ante and interim subjective beliefs.

Definition 12. The set Sub_Ω^i of *ex-ante subjective beliefs* (or *subjective priors*) of agent i at e^i is

$$\text{Sub}_\Omega^i \equiv \{p \in \text{Prob}(\Omega) : \mathbb{E}^p[x] \geq \mathbb{E}^p[e^i], \quad \forall x \in \text{Pref}_\Omega^i(e^i)\}.$$

The set Sub_π^i of *interim subjective beliefs* (or *subjective posteriors*) of agent i at e^i with private information $\pi \in \Pi^i$ is

$$\text{Sub}_\pi^i \equiv \{r \in \text{Prob}(\pi) : \mathbb{E}^r[y] \geq \mathbb{E}^r[e^i|\pi], \quad \forall y \in \text{Pref}_\pi^i(e^i|\pi)\}.$$

For any $\sigma \in \{\Omega\} \cup \Pi^i$, the set Sub_σ^i is non-empty. Indeed, the vector $e^i|_\sigma$ does not belong to the convex set $\text{Pref}_\sigma^i(e^i|_\sigma)$. Applying the Separating Hyperplane Theorem there exists a non-zero vector $\xi \in \mathbb{R}_+^\sigma$ supporting the set $\text{Pref}_\sigma^i(e^i|_\sigma)$ at $e^i|_\sigma$, i.e.,

$$\forall y \in \text{Pref}_\sigma^i(e^i|_\sigma), \quad \xi \cdot y \equiv \sum_{\omega \in \sigma} \xi(\omega)y(\omega) \geq \sum_{\omega \in \sigma} \xi(\omega)e^i(\omega) = \xi \cdot (e^i|_\sigma).$$

Fix a state $\omega \in \sigma$, $\varepsilon > 0$ and let h_ε be the vector in \mathbb{R}_{++}^σ defined by $h_\varepsilon(\omega') = \varepsilon$ for every $\omega' \neq \omega$ and $h_\varepsilon(\omega) = 1$. Since the binary relation \succ_σ^i is monotone we get $\xi \cdot h_\varepsilon \geq 0$. Passing to the limit when ε tends to 0 we can conclude that the vector ξ is a non-zero vector in \mathbb{R}_+^σ . Since ξ is not zero we can normalize ξ such that r defined by

$$\forall \omega \in \sigma, \quad r(\omega) = \frac{\xi(\omega)}{\sum_{\omega' \in \sigma} \xi(\omega')}$$

can be assimilated to a probability measure in $\text{Prob}(\sigma)$. For any vector $y \in \mathbb{R}^\sigma$, the inner product $r \cdot y$ is then denoted by $\mathbb{E}^r[y]$. Abusing notations we will assimilate the probability measure $r \in \text{Prob}(\sigma)$ to its natural extension in $\text{Prob}(\Omega)$ by posing $r(\omega) = 0$ if ω does not belong to σ .

The set Sub_σ^i is also compact and convex, therefore it belongs to $\mathcal{P}(\sigma)$. We can adapt the arguments in Rigotti, Shannon, and Strzalecki (2008) to prove the following intermediary result which is the counterpart of Lemma 6 for general convex preferences.

¹²The vector $h \in \mathbb{R}_+^\sigma$ is interior if it is strictly positive, i.e., $h(\omega) > 0$ for every $\omega \in \sigma$.

Lemma 6. Fix an agent i , a signal $\sigma \in \{\Omega\} \cup \Pi^i$ and a vector $t^i \in \mathbb{R}^\sigma$ such that $(e^i|\sigma) + t^i$ belongs to \mathbb{R}_+^σ .

- If $(e^i|\sigma) + t^i$ belongs to $\text{Pref}_\sigma^i(e^i|\sigma)$ then

$$\forall p \in \text{Sub}_\sigma^i, \quad \mathbb{E}^p[t^i] > 0. \quad (15)$$

- Reciprocally, if t^i is such that (15) is satisfied then there exists $\varepsilon > 0$ small enough such that

$$\forall \eta \in (0, \varepsilon), \quad (e^i|\sigma) + \eta t^i \in \text{Pref}_\sigma^i(e^i|\sigma).$$

For the sake of completeness, we provide a detailed proof.

Proof of Lemma 6. Assume that $(e^i|\sigma) + t^i$ belongs to $\text{Pref}_\sigma^i(e^i|\sigma)$. We propose to prove that (15) is satisfied. Since $(e^i|\sigma)$ is strictly positive and $\text{Pref}_\sigma^i(e^i|\sigma)$ is open in \mathbb{R}_+^σ , there exists $\alpha \in (0, 1]$ close enough to 1 such that $(e^i|\sigma) + \alpha t^i$ is strictly positive and still belongs to $\text{Pref}_\sigma^i(e^i|\sigma)$. Since the set $\text{Pref}_\sigma^i(e^i|\sigma)$ is open in \mathbb{R}_+^σ , there exists $\varepsilon > 0$ small enough such that¹³

$$(e^i|\sigma) + \alpha t^i - \varepsilon \mathbf{1}_\sigma \in \text{Pref}_\sigma^i(e^i|\sigma).$$

It follows from the definition of Sub_σ^i that

$$\forall p \in \text{Sub}_\sigma^i, \quad \mathbb{E}^p[t^i] \geq \frac{\varepsilon}{\alpha} > 0.$$

Assume now that $t^i \in \mathbb{R}^\sigma$ is such that $(e^i|\sigma) + t^i$ belongs to \mathbb{R}_+^σ and (15) is satisfied. Since $e^i|\sigma$ is strictly positive, there exists $\bar{\varepsilon}$ such that for every $\varepsilon \in (0, \bar{\varepsilon}]$ the vector $(e^i|\sigma) + \varepsilon t^i$ belongs to \mathbb{R}_+^σ . We claim that there exists at least one $\varepsilon \in (0, \bar{\varepsilon}]$ such that $(e^i|\sigma) + \varepsilon t^i$ belongs to $\text{Pref}_\sigma^i(e^i|\sigma)$. Assume by way of contradiction that

$$\{(e^i|\sigma) + \varepsilon t^i : \varepsilon \in (0, \bar{\varepsilon}]\} \cap \text{Pref}_\sigma^i(e^i|\sigma) = \emptyset.$$

Applying the Separating Hyperplane Theorem there exists a non-zero vector $\xi \in \mathbb{R}^\sigma$ such that

$$\forall \varepsilon \in (0, \bar{\varepsilon}], \quad \forall x \in \text{Pref}_\sigma^i(e^i|\sigma), \quad \xi \cdot x \geq \xi \cdot [(e^i|\sigma) + \varepsilon t^i].$$

Letting ε tend to 0, we obtain that ξ supports $\text{Pref}_\sigma^i(e^i|\sigma)$ at $(e^i|\sigma)$. Following a previous discussion we can prove that ξ belongs to \mathbb{R}_+^σ . Normalizing if necessary, we can assume that ξ is a subjective belief, i.e., belongs to Sub_σ^i . For each $n \in \mathbb{N}$ we let

$$x_n \equiv (e^i|\sigma) + (1/(n+1))\mathbf{1}_\sigma.$$

Since the preference relation \succsim_σ^i is monotone we have $x_n \in \text{Pref}_\sigma^i(e^i|\sigma)$ implying that¹⁴

$$\frac{1}{n+1} \geq \bar{\varepsilon} \mathbb{E}^\xi[t^i].$$

¹³The vector $\mathbf{1}_\sigma$ in \mathbb{R}^σ is defined by $\mathbf{1}_\sigma(\omega) = 1$ for every $\omega \in \sigma$.

¹⁴As usual, for any vector $z \in \mathbb{R}^\sigma$ the notation $\xi \cdot z$ is replaced by $\mathbb{E}^\xi[z]$ since ξ is a probability measure defined on σ .

Passing to the limit this leads to the contradiction: $0 \geq \mathbb{E}^\xi[t^i]$. We have thus proved that there exists $\varepsilon \in (0, \bar{\varepsilon}]$ such that $(e^i|\sigma) + \varepsilon t^i$ belongs to $\text{Pref}_\sigma^i(e^i|\sigma)$. We claim that actually $(e^i|\sigma) + \bar{\eta} t^i$ belongs to $\text{Pref}_\sigma^i(e^i|\sigma)$ for every $\bar{\eta} \in (0, \varepsilon]$. Fix $\bar{\eta} \in (0, \varepsilon]$. Following the previous argument we can prove that there exists $\eta \in (0, \bar{\eta})$ such that $(e^i|\sigma) + \eta t^i$ belongs to $\text{Pref}_\sigma^i(e^i|\sigma)$. Observe that $(e^i|\sigma) + \bar{\eta} t^i$ is a convex combination of $(e^i|\sigma) + \eta t^i$ and $(e^i|\sigma) + \varepsilon t^i$. Since the set $\text{Pref}_\sigma^i(e^i|\sigma)$ is convex, we get the desired result. \square

We adapt the concepts of efficiency to this general framework.

Definition 13. The allocation e is

- *ex-ante efficient* if there does not exist a feasible trade t such that each agent i prefers at the ex-ante stage the contingent consumption $e^i + t^i$ to e^i in the sense that $e^i + t^i \succ_\Omega^i e^i$;
- *interim efficient* if there does not exist a feasible trade t such that each agent i prefers at every interim stage $\pi \in \Pi^i$ the contingent consumption $(e^i|\pi) + (t^i|\pi)$ to $(e^i|\pi)$ in the sense that $(e^i|\pi) + (t^i|\pi) \succ_\pi^i (e^i|\pi)$.

A direct consequence of Lemma 6 and the fundamental theorem of welfare economics is the following characterization of ex-ante efficiency due to Rigotti, Shannon, and Strzalecki (2008).

Theorem 2. *The allocation e is ex-ante efficient if and only if*

$$\bigcap_{i \in I} \text{Sub}_\Omega^i \neq \emptyset.$$

Following almost verbatim the arguments in the proof of Theorem 1 we obtain the following characterization.¹⁵

Theorem 3. *The allocation e is interim efficient if and only if*

$$\bigcap_{i \in I} \text{CP}^i \left[(\text{Sub}_\pi^i)_{\pi \in \Pi^i} \right] \neq \emptyset.$$

Rigotti, Shannon, and Strzalecki (2008) studied the relationships between the notion of subjective belief and those arising in several common models of ambiguity. We propose to interpret the two previous characterization results for the two models of ambiguity studied in Kajii and Ui (2009). One could also do the same for the other models studied in Rigotti, Shannon, and Strzalecki (2008).¹⁶

¹⁵For every interim signal $\pi \in \Pi^i$, a subjective belief $r_\pi \in \text{Sub}_\pi^i$ can be interpreted as a probability measure in $\text{Prob}(\Omega)$ by posing $r_\pi(\omega) = 0$ for every $\omega \notin \pi$. Therefore, we abuse notations and consider that Sub_π^i is a subset of $\text{Prob}(\Omega)$ implying that the formula

$$\text{CP}^i \left[(\text{Sub}_\pi^i)_{\pi \in \Pi^i} \right]$$

is well-defined.

¹⁶The Bewley's incomplete preference model is not studied in Rigotti, Shannon, and Strzalecki (2008) since they restrict their attention to complete and transitive binary relations.

7.1. Bewley's incomplete preferences

In this section we consider that each agent i 's preferences are defined as follows:

- ex-ante,

$$\forall x \in \mathbb{R}_+^\Omega, \quad \text{Pref}_\Omega^i(x) \equiv \{y \in \mathbb{R}_+^\Omega : \forall p \in P^i, \quad \mathbb{E}^p[u^i(y)] > \mathbb{E}^p[u^i(x)]\}$$

- for every interim signal $\pi \in \Pi^i$,¹⁷

$$\forall x \in \mathbb{R}_+^\pi, \quad \text{Pref}_\pi^i(x) \equiv \{y \in \mathbb{R}_+^\pi : \forall r \in \Phi^i(\pi), \quad \mathbb{E}^r[u^i(y)] > \mathbb{E}^r[u^i(x)]\}.$$

For these specific convex preferences we can compute explicitly the set of subjective beliefs.

Lemma 7. *For each agent i we have*

$$\text{Sub}_\Omega^i = \text{RA}^i(P^i) \quad \text{and} \quad \text{Sub}_\pi^i = \text{RA}^i(\Phi^i(\pi)), \quad \forall \pi \in \Pi^i.$$

The arguments of the proof are standard: the result follows from the concavity of the utility index u^i . As a direct consequence of Theorem 2, Theorem 3 and the previous lemma, we obtain the following necessary and sufficient conditions for ex-ante and interim efficiency.

Proposition 8. Assume that all agents have Bewley's incomplete preferences. The allocation e is ex-ante efficient if and only if

$$\bigcap_{i \in I} \text{RA}^i(P^i) \neq \emptyset.$$

Proposition 8 corresponds to Proposition 1 in Kajii and Ui (2009) which is due to Bewley (2001) and Rigotti and Shannon (2005).

Proposition 9. Assume that all agents have Bewley's incomplete preferences. The allocation e is interim efficient if and only if

$$\bigcap_{i \in I} \text{CP}^i \circ \text{RA}^i(\Phi^i) \neq \emptyset.$$

Proposition 9 corresponds to Proposition 2 in Kajii and Ui (2009).¹⁸

¹⁷If x is a vector in \mathbb{R}_+^π and r is a probability in $\text{Prob}(\pi)$ we let $\mathbb{E}^r[u^i(x)] \equiv \sum_{\omega \in \pi} r(\omega)u^i(x(\omega))$.

¹⁸Actually in Kajii and Ui (2009) the necessary and sufficient condition for interim efficiency is

$$\bigcap_{i \in I} \text{RA}^i \circ \text{CP}^i(\Phi^i) \neq \emptyset.$$

We proved (see Lemma 3) that this condition is equivalent to the one proposed in Proposition 9.

7.2. Gilboa–Schmeidler’s MaxMin expected utility preferences

In this section we consider that each agent i ’s preferences are defined as follows:

- ex-ante,

$$\forall x \in \mathbb{R}_+^\Omega, \quad \text{Pref}_\Omega^i(x) \equiv \left\{ y \in \mathbb{R}_+^\Omega : \min_{p \in P^i} \mathbb{E}^p[u^i(y)] > \min_{p \in P^i} \mathbb{E}^p[u^i(x)] \right\}$$

- for every interim signal $\pi \in \Pi^i$,

$$\forall x \in \mathbb{R}_+^\pi, \quad \text{Pref}_\pi^i(x) \equiv \left\{ y \in \mathbb{R}_+^\pi : \min_{r \in \Phi^i(\pi)} \mathbb{E}^r[u^i(y)] > \min_{r \in \Phi^i(\pi)} \mathbb{E}^r[u^i(x)] \right\}.$$

For these specific convex preferences Rigotti, Shannon, and Strzalecki (2008) have computed explicitly the set of subjective beliefs.¹⁹

Lemma 8. *For each agent i we have*

$$\text{Sub}_\Omega^i = \text{RA}^i \circ \text{Act}^i(P^i) \quad \text{and} \quad \text{Sub}_\pi^i = \text{RA}^i \circ \text{Act}^i(\Phi^i(\pi)), \quad \forall \pi \in \Pi^i.$$

As a direct consequence of Theorem 2, Theorem 3 and the previous lemma, we obtain the necessary and sufficient conditions for ex-ante and interim efficiency presented in Proposition 1 and Theorem 1.

8. Interim efficiency and common knowledge

In this section we consider the framework of the previous section where each agent i is endowed with an ex-ante preference relation \succ_Ω^i and an interim preference relation \succ_π^i for every private signal $\pi \in \Pi^i$. We assume that Assumption 3 is satisfied, i.e., for each $\sigma \in \{\Omega\} \cup \Pi^i$, the preference relation \succ_σ^i is irreflexive, convex, monotone and continuous.

The no-trade theorem of Milgrom and Stokey (1982) says that, upon the new arrival of information to the agents, it cannot be commonly known among them that there are some trading opportunities which can make them mutually beneficial, provided that the previous allocation (before the arrival of the new information) is ex-ante efficient. To illustrate this we consider that the allocation e is the outcome of an ex-ante trade process and assume it is ex-ante efficient. If the state of nature is $s \in \Omega$, each agent i knows (and only knows) at the interim stage that the true state belongs to $\pi^i \equiv \Pi^i(s)$. One should first define which objects agents may have incentive to trade. If agent i proposes to trade a consumption bundle $x_{\pi^i} : \pi^i \rightarrow \mathbb{R}_+$ he is revealing to the other agents his private signal π^i . We follow Wilson (1978) by considering that agents do not want to communicate their private information. If we assume that the family $(\Pi^i)_{i \in I}$ of private partitions is common knowledge then agents can trade consumption bundles contingent to the common knowledge event $E \equiv \Pi^c(s)$ where $\Pi^c(s)$ is the unique atom of the common knowledge partition Π^c .²⁰ Let $(y_E^i)_{i \in I}$ be an allocation of consumption bundles $y_E^i \in \mathbb{R}_+^E$ contingent to the event E and feasible, i.e.,

$$\forall \omega \in E, \quad \sum_{i \in I} y_E^i(\omega) = \sum_{i \in I} e^i(\omega).$$

¹⁹See Lemma 1 in Rigotti, Shannon, and Strzalecki (2008) and Appendix A in Kajii and Ui (2009).

²⁰ Π^c is the finest partition among those which are coarser than all partitions $\{\Pi^i : i \in I\}$. See Aumann (1976).

When would agents accept to exchange the allocation $(e^i)_{i \in I}$ with the allocation $(y_E^i)_{i \in I}$? Agent i would accept if the restriction $(y_E^i | \pi^i)$ is strictly preferred to $(e^i | \pi^i)$, i.e.,

$$(y_E^i | \pi^i) \succ_{\pi^i}^i (e^i | \pi^i) \quad \text{or equivalently} \quad (y_E^i | \pi) \in \text{Pref}_{\pi}^i(e^i | \pi)$$

or if there is status-quo, i.e.,

$$(y_E^i | \pi^i) = (e^i | \pi^i).$$

We denote by $\Pi^i(E)$ the set of all atoms $\pi \in \Pi^i$ contained in E . The set $\Pi^i(E)$ is the collection of all signals agent i may have received according to the common knowledge information. Assuming that each agents' characteristics are common knowledge, agent i can accept to trade only if for every signal $\pi \in \Pi^i(E)$ we also have

$$(y_E^i | \pi) \in \{(e^i | \pi)\} \bigcup \text{Pref}_{\pi}^i(e^i | \pi). \quad (16)$$

Indeed, if agent i accepts to trade despite the fact that there exists $\pi \in \Pi^i(E)$ such that (16) is not satisfied then the other agents would infer that the true state does not belong to π . Since we assume that each agent does not want to reveal information to the others, we are lead to consider the following concept of interim efficiency.

Definition 14. The allocation e is *status-quo (SQ-)interim efficient* if there does not exist a feasible allocation $(y^i)_{i \in I}$ such that for each agent i and for every signal $\pi \in \Pi^i$, we have

$$(y^i | \pi) \in \{(e^i | \pi)\} \bigcup \text{Pref}_{\pi}^i(e^i | \pi)$$

and for at least one agent j and one signal $\pi^j \in \Pi^j$, we have

$$(y^j | \pi^j) \in \text{Pref}_{\pi^j}^j(e^j | \pi^j).$$

Remark 3. One may define the concept of status-quo ex-ante efficiency as follows: the allocation e is status-quo ex-ante efficient if there does not exist a feasible allocation $(y^i)_{i \in I}$ such that for each agent i , we have

$$y^i \in \{e^i\} \bigcup \text{Pref}_{\Omega}^i(e^i)$$

and for at least one agent j we have

$$y^j \in \text{Pref}_{\Omega}^j(e^j).$$

If an allocation is status-quo ex-ante efficient then it is ex-ante efficient. The converse is also true since ex-ante preferences are monotone and continuous.²¹

²¹Let $(y^i)_{i \in I}$ be a feasible allocation such that $y^i = e^i$ or $y^i \succ_{\Omega}^i e^i$ for each agent i and $y^j \succ_{\Omega}^j e^j$ for some agent j . Since preferences are continuous, there exists α close enough to 1 such that the allocation $(z^i)_{i \in I}$ defined by $z^i \equiv \alpha y^i + (1 - \alpha)e^i$ is feasible, interior and satisfies $z^i = e^i$ or $z^i \succ_{\Omega}^i e^i$ for each agent i and $z^j \succ_{\Omega}^j e^j$. Since the binary relation \succ_{Ω}^j is continuous there exists $\varepsilon > 0$ small enough such that $x^j \equiv z^j - \varepsilon \mathbf{1}_{\Omega} \succ_{\Omega}^j e^j$. For every agent $i \neq j$ we pose $x^i \equiv z^i + \beta \mathbf{1}_{\Omega}$ where $\beta(\#I - 1) = \varepsilon$. Observe that the allocation $(x^i)_{i \in I}$ is feasible and satisfies $x^i \succ_{\Omega}^i e^i$ for each agent i .

If the allocation e is status-quo interim efficient then it is interim efficient. The converse is not true as illustrated by the following example.

Example 3. Consider an exchange economy with three states of nature $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and two risk-neutral agents $I = \{i_1, i_2\}$ with MEU-preferences. Each agent i may receive two signals $\Pi^i = \{\alpha, \beta\}$ with $\alpha = \{\omega_1, \omega_2\}$ and $\beta = \{\omega_3\}$. Observe that independent of agents' posterior beliefs $\Phi^i(\alpha)$ the allocation $(e^i)_{i \in I}$ is interim efficient (as defined by Kajii and Ui (2009)).²² However, one can choose the allocation $(e^i)_{i \in I}$ and agents' posterior beliefs such that agents have incentives to trade at the interim stage if the state of nature is ω_1 or ω_2 .

In most models with ambiguity aversion, the strict preference relation \succ_{π}^i of agent i contingent to signal π is the strict part of a reflexive, complete and transitive binary relation \succeq_{π}^i .

Definition 15. Preferences are said *complete, transitive and convex* if for every agent i and every signal $\sigma \in \{\Omega\} \cup \Pi^i$, the binary relation \succ_{σ}^i satisfies Assumption 3 and is the strict part of a reflexive, complete, transitive and convex binary relation \succeq_{σ}^i .²³

When preferences are complete transitive and convex one may consider the concept of interim efficiency used by Milgrom and Stokey (1982) and Morris (1994).

Definition 16. Assume that preferences are complete transitive and convex. The allocation e is *M-interim efficient* if there does not exist a feasible allocation $(y^i)_{i \in I}$ such that for each agent i and for every signal $\pi \in \Pi^i$, we have

$$(y^i | \pi) \succeq_{\pi}^i (e^i | \pi)$$

and for at least one agent j and one signal $\pi^j \in \Pi^j$, we have

$$(y^j | \pi^j) \succ_{\pi^j}^j (e^j | \pi^j).$$

Remark 4. Assume that preferences are complete transitive and convex. If the allocation e is M-interim efficient then it is status-quo interim efficient. The converse is true if preferences are strictly convex in the sense that for every agent $i \in I$, for every signal $\pi \in \Pi^i$ and every consumption bundle $z_{\pi} \in \mathbb{R}_{+}^{\pi}$,

$$z_{\pi} \succeq_{\pi}^i (e^i | \pi) \implies \alpha z_{\pi} + (1 - \alpha)(e^i | \pi) \succ_{\pi}^i (e^i | \pi), \quad \forall \alpha \in (0, 1).$$

Remark 5. When preferences are complete transitive and convex then the set of subjective beliefs coincide with the set of probabilities (or prices) supporting upper contour sets as defined by Yaari (1969) and Rigotti, Shannon, and Strzalecki (2008). More precisely, for every agent i and every signal $\sigma \in \{\Omega\} \cup \Pi^i$, the vector p is a subjective belief if and only if $\mathbb{E}^p[y] \geq \mathbb{E}^p[x]$ for every $y \succeq_{\sigma}^i x$.

²²Indeed, assume by way of contradiction that there exists a feasible allocation $(y^i)_{i \in I}$ such that $(y^i | \pi) \succ_{\pi}^i (e^i | \pi)$ for every agent i and every signal $\pi \in \Pi^i$. Then one must have $y^i(\omega_3) > e^i(\omega_3)$ for each i . This contradicts the feasibility of $(y^i)_{i \in I}$.

²³We omit the standard definition of a reflexive, complete and transitive binary relation \succeq_{σ}^i . The binary relation \succeq_{σ}^i is said convex when $\{y \in \mathbb{R}_{+}^{\sigma} : y \succeq_{\sigma}^i x\}$ is convex for every $x \in \mathbb{R}_{+}^{\sigma}$.

We proved that the existence of a common prior consistent with subjective posteriors is a necessary and sufficient condition for interim efficiency. To characterize SQ-interim and M-interim efficiency, we need to strengthen this condition.

Definition 17. Fix an agent i and a family $Q = (Q(\pi))_{\pi \in \Pi^i}$ of posterior beliefs where for each signal $\pi \in \Pi^i$, the set $Q(\pi)$ belongs to $\mathcal{P}(\pi)$. A probability $p \in \text{Prob}(\Omega)$ is said to be *fully consistent* with posteriors of Q if for every signal $\pi \in \Pi^i$ we have $p(\pi) > 0$ and the conditional probability $p(\cdot|\pi)$ belongs to the set $Q(\pi)$. The set of all priors fully consistent with posteriors of Q is denoted by

$$\text{CP}_+^i(Q).$$

Remark 6. A prior p is consistent with posteriors of Q if there exists $(\lambda_\pi, r_\pi)_{\pi \in \Pi^i}$ with $r_\pi \in Q(\pi)$ and λ a vector in $\text{Prob}(\Pi^i)$ such that

$$p = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi.$$

The prior p is fully consistent with posteriors of Q if λ has a full support, i.e., $\lambda_\pi > 0$ for each possible signal $\pi \in \Pi^i$.

It is straightforward to check that existence of a common prior that is fully consistent with subjective posteriors is a sufficient condition for SQ-interim efficiency and M-interim efficiency (when defined).²⁴ This sufficient condition turns out to be necessary if interim preferences are strictly monotone.

Definition 18. Interim preferences are said to be *strictly monotone* if for every agent i and every signal $\pi \in \Pi^i$ we have $x + h \succ_\pi^i x$ for every consumption bundles $x, h \in \mathbb{R}_+^\pi$ where $h \neq 0$.

Observe that interim MEU-preferences are strictly monotone if every posterior belief has full support in the sense that for every agent i , for every signal $\pi \in \Pi^i$, every posterior $r \in \Phi^i(\pi)$ satisfies $\text{supp } r = \pi$.²⁵

Theorem 4. Assume that interim preferences are strictly monotone. Existence of a common prior that is fully consistent with subjective posteriors is a necessary and sufficient condition for SQ-interim efficiency (or M-interim efficiency when preferences are complete transitive and convex).

In other words, when interim preferences are strictly monotone, the allocation e is SQ-interim efficient (or M-interim efficient when defined) if and only if

$$\bigcap_{i \in I} \text{CP}_+^i \left[(\text{Sub}_\pi^i)_{\pi \in \Pi^i} \right] \neq \emptyset.$$

We postpone the details of the proof to Appendix D. As a consequence of the previous theorem, we obtain the following characterization of SQ-interim efficiency (and thus M-interim efficiency).

Proposition 10. Assume that interim preferences are strictly monotone. The allocation e is SQ-interim efficient if and only if for every common knowledge atom $E \in \Pi^c$ the allocation $(e^i|E)_{i \in I}$ is interim efficient (as defined by Kajii and Ui (2009)) when the state space is restricted to E .

²⁴See Proposition 12 in Appendix D.

²⁵If $r \in \text{Prob}(\Omega)$ we denote by $\text{supp } r$ the support of r defined by $\text{supp } r \equiv \{\omega \in \Omega : r(\omega) > 0\}$.

9. M-speculative trade

For expositional reasons, we assume in this section that agents have MEU-preferences satisfying the assumptions of Section 2. We consider another concept of speculative trade by replacing the interim efficiency by the M-interim efficiency.

Definition 19. We say that there is no *M-speculative trade* if ex-ante efficiency of the allocation e implies that it is also M-interim efficient.

The concept of M-speculative trade corresponds to the one used in Milgrom and Stokey (1982) and Morris (1994). The following example illustrates the differences with the speculative trade concept used by Kajii and Ui (2009).

Example 4. We consider the economy defined in Example 3. We also assume that: contingent to the signal $\alpha = \{\omega_1, \omega_2\}$ each agent i has a single posterior r_α^i , endowments are privately measurable and priors are rectangular with respect to posteriors. Priors are defined as follows:

$$P^i = \Lambda \otimes \Phi^i$$

where $\Lambda = \{\mu, \eta\}$ is a set of two probability measures in $\text{Prob}(\Pi)$ defined by $\mu_\alpha = 1$ and $\eta_\beta = 1$. In other words, each agent considers that there are two scenarios. If the first (second) scenario is correct then each agent will receive at the interim stage the signal α (resp. β). Observe that there is no speculative trade since Corollary 1 and Corollary 2 apply. However, if $e^i(\omega_3) < e^i(\omega_2)$ then the allocation e is ex-ante efficient but if the posteriors r^{i_1} and r^{i_2} are distinct with full support in α then e is not M-interim efficient.²⁶

The previous example shows that Corollary 1 and Corollary 2 are not valid if we replace “speculative trade” by “M-speculative trade”. However, it is straightforward to adapt Proposition 7 in order to obtain a sufficient condition for no M-speculative trade.

Proposition 11. Assume that for each agent i and each active prior $p^i \in \text{Act}^i(P^i)$, every signal $\pi \in \Pi^i$ is plausible and the conditional probability $p^i(\cdot|\pi)$ is an active posterior, i.e.,

$$\forall p^i \in \text{Act}^i(P^i), \quad \forall \pi \in \Pi^i, \quad p^i(\pi) > 0 \quad \text{and} \quad p^i(\cdot|\pi) \in \text{Act}^i(\Phi^i(\pi)). \quad (17)$$

Then there is no M-speculative trade.

Remark 7. We do not need to assume that interim preferences are strictly monotone for the above proposition to be valid. Indeed, the existence of a common prior fully consistent with subjective posteriors is a sufficient condition for M-interim efficiency even if interim preferences are not strictly monotone (see Proposition 12 in Appendix refapp:M-interim-efficiency).

²⁶Since e^i is privately measurable, we denote by $e^i(\alpha)$ the constant value of e^i in α . Observe that $P^i = \{r_\alpha^i, \mathbf{1}_\beta\}$ implying that

$$\mathbb{E}^{r_\alpha^i}[e^i] = e^i(\alpha) > e^i(\omega_3) = \mathbb{E}^{\mathbf{1}_\beta}[e^i].$$

It follows that $\text{Act}^i(P^i) = \{\mathbf{1}_\beta\}$ and the allocation e is ex-ante efficient. Now assume that e is M-interim efficient. Since interim preferences are strictly increasing we can apply Theorem 4 to get the existence of a common prior $p \in \text{Prob}(\Omega)$ that is fully consistent with posteriors, i.e.,

$$p = p(\alpha)r_\alpha^i + p(\beta)\mathbf{1}_\beta \quad \text{with} \quad p(\alpha) > 0 \quad \text{and} \quad p(\beta) > 0.$$

As a consequence, we obtain the following contradiction: $r_\alpha^{i_1} = r_\alpha^{i_2}$.

Observe that if ex-ante preferences are strictly monotone then any active prior $p^i \in \text{Act}^i(P^i)$ will assign positive probability to any signal $\pi \in \Pi^i$. Actually we do not need to assume that preferences are strictly monotone.

Definition 20. Ex-ante preferences are said *signal-monotone* if for any contingent consumption bundles x and h in \mathbb{R}_+^Ω we have $x + h \succ_\Omega^i x$ when there exists at least one signal $\pi \in \Pi^i$ such that h is strictly positive on π , i.e., $h(\omega) > 0$ for every $\omega \in \pi$.

A straightforward consequence of Proposition 11 is the following counter-part of Corollary 1.

Corollary 3. Assume that ex-ante preferences are signal-monotone and the set of posteriors is Bayesian consistent with the set of priors. If the allocation e is privately measurable then there is no M -speculative trade.

Signal-monotonicity is automatically satisfied when priors are fully rectangular with respect to posteriors as defined below.

Definition 21. The set of priors is *fully rectangular with respect to posteriors* if for each agent i we have $P^i = \Lambda^i \otimes \Phi^i$ where the set Λ^i of beliefs in $\text{Prob}(\Pi^i)$ is such that every $\lambda \in \Lambda^i$ assigns positive probability $\lambda_\pi > 0$ to every signal $\pi \in \Pi^i$.

A straightforward consequence of Proposition 11 is the following counter-part of Corollary 2.

Corollary 4. If for each agent the set of priors is fully rectangular with respect to posteriors, then there is no M -speculative trade.

Observe that the set of priors in Example 4 are rectangular with respect to posteriors but not fully rectangular.

Appendix

A. Proof of Lemma 3

We first prove that $\text{CP}^i \circ \text{RA}^i(Q) \subset \text{RA}^i \circ \text{CP}^i(Q)$.

Proof. Let $q \in \text{CP}^i \circ \text{RA}^i(Q)$. There exist $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ a vector in $\text{Prob}(\Pi^i)$ and a family $(r_\pi)_{\pi \in \Pi^i}$ such that

$$q = \sum_{\pi \in \Pi^i} \lambda_\pi \text{RA}^i(r_\pi) \quad \text{and} \quad \forall \pi \in \Pi^i, \quad r_\pi \in Q(\pi).$$

For each state $\omega \in \Omega$, we have

$$\begin{aligned} q(\omega) &= \sum_{\pi \in \Pi^i} \frac{\lambda_\pi}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]} r_\pi(\omega) \nabla u^i(e^i(\omega)) \\ &= \alpha \sum_{\pi \in \Pi^i} \mu_\pi r_\pi(\omega) \nabla u^i(e^i(\omega)) \end{aligned}$$

where

$$\alpha \equiv \sum_{\pi \in \Pi^i} \frac{\lambda_\pi}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]} \quad \text{and} \quad \forall \pi \in \Pi^i, \quad \mu_\pi \equiv \frac{1}{\alpha} \frac{\lambda_\pi}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]}.$$

We let p be defined by

$$p \equiv \sum_{\pi \in \Pi^i} \mu_\pi r_\pi.$$

Observe that p belongs to $\text{CP}^i(Q)$ and $q = \beta \text{RA}^i(p)$ where $\beta \equiv \alpha \mathbb{E}^p[\nabla u^i(e^i)]$. To finish the proof it is sufficient to show that β is equal to 1. Since for each signal $\pi \in \Pi^i$, the support of r_π is a subset of π , we have

$$\begin{aligned} \mathbb{E}^p[\nabla u^i(e^i)] &= \sum_{\omega \in \Omega} \left[\sum_{\pi \in \Pi^i} \mu_\pi r_\pi(\omega) \right] \nabla u^i(e^i(\omega)) \\ &= \sum_{\pi \in \Pi^i} \mu_\pi \sum_{\omega \in \pi} r_\pi(\omega) \nabla u^i(e^i(\omega)) \\ &= \sum_{\pi \in \Pi^i} \mu_\pi \mathbb{E}^{r_\pi}[\nabla u^i(e^i)] \\ &= \sum_{\pi \in \Pi^i} \frac{\lambda_\pi}{\alpha} = \frac{1}{\alpha}. \end{aligned}$$

We have thus proved that $q = \text{RA}^i(p)$ where p belongs to $\text{CP}^i(Q)$. □

Now, we prove that $\text{RA}^i \circ \text{CP}^i(Q) \subset \text{CP}^i \circ \text{RA}^i(Q)$.

Proof. Fix $p \in \text{CP}^i(Q)$. There exist $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ a vector in $\text{Prob}(\Pi^i)$ and a family $(r_\pi)_{\pi \in \Pi^i}$ such that

$$p = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi \quad \text{and} \quad \forall \pi \in \Pi^i, \quad r_\pi \in Q(\pi).$$

Now let q be the risk-adjusted prior $\text{RA}^i(p)$. For each state $\omega \in \Omega$, we have

$$\begin{aligned} q(\omega) &= \frac{1}{\mathbb{E}^p[\nabla u^i(e^i)]} \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi(\omega) \nabla u^i(e^i(\omega)) \\ &= \sum_{\pi \in \Pi^i} \mu_\pi \frac{1}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]} r_\pi(\omega) \nabla u^i(e^i(\omega)) \end{aligned}$$

where

$$\mu_\pi = \frac{\lambda_\pi \times \mathbb{E}^{r_\pi}[\nabla u^i(e^i)]}{\mathbb{E}^p[\nabla u^i(e^i)]}.$$

Since the vector $(\mu_\pi)_{\pi \in \Pi^i}$ belongs to $\text{Prob}(\Pi^i)$, we obtain that

$$q = \sum_{\pi \in \Pi^i} \mu_\pi \text{RA}^i(r_\pi)$$

and therefore q belongs to $\text{CP}^i \circ \text{RA}^i(Q)$. □

B. Proof of Lemma 4

We first prove that $\text{Act}^i \circ \text{CP}^i(Q) \subset \text{CP}^i \circ \text{Act}^i(Q)$.

Proof. Fix $p \in \text{Act}^i \circ \text{CP}^i(Q)$. There exist $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ a vector in $\text{Prob}(\Pi^i)$ and a family $(r_\pi)_{\pi \in \Pi^i}$ such that

$$p = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi \quad \text{and} \quad \forall \pi \in \Pi^i, \quad r_\pi \in Q(\pi).$$

It is sufficient to show that each r_σ actually belongs to $\text{Act}^i(Q(\sigma))$ for every signal $\sigma \in \Pi^i$ satisfying $\lambda_\sigma > 0$. Fix a signal σ with $\lambda_\sigma > 0$ and any posterior $q_\sigma \in Q(\sigma)$. We let p^σ be the probability in $\text{Prob}(\Omega)$ defined by

$$p^\sigma \equiv \lambda_\sigma q_\sigma + \sum_{\pi \neq \sigma} \lambda_\pi r_\pi.$$

The probability measure p^σ belongs to $\text{CP}^i(Q)$ by construction. It follows that

$$\mathbb{E}^p[u^i(e^i)] \leq \mathbb{E}^{p^\sigma}[u^i(e^i)]$$

implying that

$$\mathbb{E}^{r_\sigma}[u^i(e^i)] \leq \mathbb{E}^{q_\sigma}[u^i(e^i)].$$

We have thus proved that r_σ belongs to $\text{Act}^i(Q(\sigma))$. \square

Now we prove that if the utility at an interim stage of e^i is independent of the signal received then $\text{CP}^i \circ \text{Act}^i(Q) \subset \text{Act}^i \circ \text{CP}^i(Q)$.

Proof. Let $q \in \text{CP}^i \circ \text{Act}^i(Q)$. There exist $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ a vector in $\text{Prob}(\Pi^i)$ and a family $(r_\pi)_{\pi \in \Pi^i}$ such that

$$q = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi \quad \text{and} \quad \forall \pi \in \Pi^i, \quad r_\pi \in \text{Act}^i[Q(\pi)].$$

Since the function $\pi \mapsto \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i)]$ is constant on Π^i , we denote its value by $u^i(\Pi^i)$. Observe that

$$\mathbb{E}^q[u^i(e^i)] = \sum_{\pi \in \Pi^i} \lambda_\pi \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i)] = u^i(\Pi^i).$$

Now, fix $\eta \in \text{CP}^i(Q)$. There exist $\gamma = (\gamma_\pi)_{\pi \in \Pi^i}$ a vector in $\text{Prob}(\Pi^i)$ and a family $(\kappa_\pi)_{\pi \in \Pi^i}$ such that

$$\eta = \sum_{\pi \in \Pi^i} \gamma_\pi \kappa_\pi \quad \text{and} \quad \forall \pi \in \Pi^i, \quad \kappa_\pi \in Q(\pi).$$

Observe that

$$\begin{aligned} \mathbb{E}^q[u^i(e^i)] = u^i(\Pi^i) &= \sum_{\pi \in \Pi^i} \gamma_\pi \mathbb{E}_{Q(\pi)}[u^i(e^i)] \\ &\leq \sum_{\pi \in \Pi^i} \gamma_\pi \mathbb{E}^{\kappa_\pi}[u^i(e^i)] \\ &\leq \mathbb{E}^\eta[u^i(e^i)]. \end{aligned}$$

We have thus proved that q belongs to $\text{Act}^i \circ \text{CP}^i(Q)$. \square

C. Proof of Lemma 5

Let p be a probability measure in $\text{CP}^i(\Phi^i)$. There exist a vector $(\lambda_\pi)_{\pi \in \Pi^i}$ in $\text{Prob}(\Pi^i)$ and a posterior belief $r_\pi \in \Phi^i(\pi)$ for each signal $\pi \in \Pi^i$ such that

$$p = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi.$$

We denote by q the risk adjusted probability $\text{RA}^i(p)$. It follows that for every state $\omega \in \Omega$,

$$\begin{aligned} q(\omega) &= \frac{1}{\mathbb{E}^p[\nabla u^i(e^i)]} \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi(\omega) \nabla u^i(e^i(\omega)) \\ &= \frac{1}{\mathbb{E}^p[\nabla u^i(e^i)]} \sum_{\pi \in \Pi^i} \nabla u^i(e^i(\pi)) \lambda_\pi r_\pi(\omega) \\ &= \sum_{\pi \in \Pi^i} \gamma_\pi r_\pi(\omega) \end{aligned}$$

where $\gamma_\pi \equiv \nabla u^i(e^i(\pi)) \lambda_\pi / \mathbb{E}^p[\nabla u^i(e^i)]$. Observe that

$$\mathbb{E}^p[\nabla u^i(e^i)] = \sum_{\pi \in \Pi^i} \lambda_\pi \mathbb{E}^{r_\pi}[u^i(e^i)] = \sum_{\pi \in \Pi^i} \lambda_\pi \nabla u^i(e^i(\pi)).$$

This implies that the vector $(\gamma_\pi)_{\pi \in \Pi^i}$ belongs to $\text{Prob}(\Pi^i)$ and therefore the risk-adjusted Φ^i -compatible prior q is also a Φ^i -compatible prior, i.e., $q \in \text{CP}^i(\Phi^i)$. We have thus proved that $\text{RA}^i \circ \text{CP}^i(\Phi^i) \subset \text{CP}^i(\Phi^i)$.

Conversely, we can always write p in the following form

$$p = \sum_{\pi \in \Pi^i} \frac{1}{\mathbb{E}^p[\nabla u^i(e^i)]} \eta_\pi \nabla u^i(e^i(\pi)) r_\pi$$

where $\eta_\pi \equiv \lambda_\pi \mathbb{E}^p[\nabla u^i(e^i)] / \nabla u^i(e^i(\pi))$. Since the vector $(\eta_\pi)_{\pi \in \Pi^i}$ belongs to $\text{Prob}(\Pi^i)$ we get that p is also a risk-adjusted Φ^i -compatible prior. We have thus proved that $\text{CP}^i(\Phi^i) \subset \text{RA}^i \circ \text{CP}^i(\Phi^i)$.

D. Proof of Theorem 4

We only provide the details of the characterization of M-interim efficiency. It is straightforward to adapt these arguments to characterize SQ-interim efficiency. We split the proof in two steps. We first prove that the condition is sufficient.

Proposition 12. Assume that there exists a common prior that is fully consistent with subjective posteriors. Then the allocation e is M-interim efficient.

Proof of Proposition 12. Assume that there exists a probability $p \in \text{Prob}(\Omega)$ such that for each agent i and every signal $\pi \in \Pi^i$, we have $p(\pi) > 0$ and there exists a subjective belief $r_\pi^i \in \text{Sub}_\pi^i$ satisfying

$$\forall i \in I, \quad p = \sum_{\pi \in \Pi^i} p(\pi) r_\pi^i.$$

Assume by way of contradiction that the allocation e is not M-interim efficient. Then there exists a feasible allocation $(y^i)_{i \in I}$ such that for each agent i and every signal $\pi \in \Pi^i$, we have

$$(y^i|\pi) \succeq_{\pi}^i (e^i|\pi)$$

and for at least one agent k and one signal $\sigma \in \Pi^k$, we have

$$(y^k|\sigma) \succ_{\sigma}^k (e^k|\sigma).$$

Since r_{σ}^k is a subjective belief we have

$$\mathbb{E}^{r_{\sigma}^k}[y^k] > \mathbb{E}^{r_{\sigma}^k}[e^k]. \quad (18)$$

It follows from Remark 5 that every probability r_{π}^i is supporting the upper contour of $(e^i|\pi)$, i.e.,

$$\mathbb{E}^{r_{\pi}^i}[y^i] \geq \mathbb{E}^{r_{\pi}^i}[e^i]. \quad (19)$$

Combining (18) and (19) we get the following contradiction

$$\sum_{i \in I} \mathbb{E}^p[y^i] = \sum_{i \in I} \sum_{\pi \in \Pi^i} p(\pi) \mathbb{E}^{r_{\pi}^i}[y^i] > \sum_{i \in I} \sum_{\pi \in \Pi^i} p(\pi) \mathbb{E}^{r_{\pi}^i}[e^i] = \sum_{i \in I} \mathbb{E}^p[e^i].$$

□

We now propose to prove that the condition is necessary.

Proposition 13. Assume that allocation e is M-interim efficient. Then there exists a common prior that is fully consistent with subjective posteriors.

Proof of Proposition 13. Fix E an element of the common knowledge partition Π^c . Since the allocation e is M-interim efficient it is also M-interim efficient when the state space is restricted to E . In particular it is interim efficient when the state space is restricted to E . Applying Theorem 3 there exists a probability $p_E \in \text{Prob}(E)$ such that for each agent i and each signal π compatible with E , i.e., $\pi \in \Pi^i(E)$, there exists a subjective belief $r_{\pi}^i \in \text{Sub}_{\pi}^i$ satisfying

$$\forall i \in I, \quad p_E = \sum_{\pi \in \Pi^i(E)} p_E(\pi) r_{\pi}^i. \quad (20)$$

Fix an agent $k \in I$. Since p_E belongs to $\text{Prob}(E)$ there exists at least one signal $\sigma \in \Pi^k(E)$ such that $p_E(\sigma) > 0$. We have to prove that $p_E(\pi) > 0$ for every agent i and every signal $\pi \in \Pi^i(E)$. Using the strict monotonicity of preferences we obtain the following intermediary step.

Claim 1. Fix an agent i and a signal $\pi \in \Pi^i(E)$ consistent with σ in the sense that $\pi \cap \sigma \neq \emptyset$. Then $p_E(\pi) > 0$.

Proof of Claim 1. Since the preference relation \succ_{σ}^k is strictly increasing the subjective belief r_{σ}^k has full support, i.e., $\text{supp } r_{\sigma}^k = \sigma$. Since $p_E(\sigma) > 0$ we obtain that $p_E(\omega) > 0$ for every $\omega \in \sigma$, implying the desired result. □

Now fix an agent j and a signal $\mu \in \Pi^j(E)$. Since E is an element of the common knowledge partition Π^c there exists a finite chain

$$((i_1, \pi_1), \dots, (i_n, \pi_n))$$

such that $(i_1, \pi_1) = (k, \sigma)$, $(i_n, \pi_n) = (j, \mu)$ and

$$\forall s \in \{1, 2, \dots, n-1\}, \quad \pi_s \cap \pi_{s+1} \neq \emptyset.$$

Applying recursively Claim 1 we get that $p_E(\mu) = p_E(\pi_n) > 0$.

Since Π^c is a partition of Ω , we let p be the probability defined by $p(\omega) = p_E(\omega)$ where E is the unique atom in Π^c containing ω . The probability p is a common prior that is fully consistent with subjective posteriors. \square

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